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Higher-dimensional AdS waves and *pp*-waves with conformally related sources

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AdS waves and *pp*-waves can only be supported by pure radiation fields, for which the only nonvanishing component of the energy-momentum tensor is the energy density along the retarded time. We show that the nonminimal coupling of self-gravitating scalar fields to the higher-dimensional versions of these exact gravitational waves can be done consistently. In both cases, the resulting *pure radiation constraints* completely fix the scalar field dependence and the form of the allowed self-interactions. More significantly, we establish that the two sets of pure radiation constraints are conformally related for any nonminimal coupling, in spite of the fact that the involved gravitational fields are not necessarily related. In this correspondence, the potential supporting the AdS waves emerges from the self-interaction associated to the *pp*-waves and a self-dual condition naturally satisfied by the *pp*-wave scalar fields.

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I. INTRODUCTION

In the current literature, there exists a wide class of examples for which the use of conformal symmetries or techniques has been useful to understand and to solve very specific problems. For example, conformal transformations of metrics can be used as a mathematical tool to map the equations of motion of physical systems into equivalently equations that are more simple to analyze. From the physical perspective this transformation entails a change of frame, and in diverse contexts some quantities only acquire a clear physical interpretation in definite frames. In this spirit, the derivation of the Bekenstein black hole provided an interesting illustration [1]. Certainly, in this case the mapping has been operated between a conformal scalar field and a minimally coupled one, and has permitted to generate a black hole solution for the conformal theory from the singular solution of the minimal scalar field theory. It is also natural to ask wether spacetime metrics which are conformally related may share some properties, beyond those generated from their common causal structure for regular conformal factors. For example, in arbitrary dimension D the Siklos spacetimes [2], which are exact gravitational waves traveling along AdS_D [3],[1] can be defined as a conformal

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[1] See Refs. [4, 5, 6, 7] for the pioneering works studying the propagation of exact gravitational waves in presence of a cosmological constant, for a review on this subject see Ref. [8].

transformation of a pp-wave background in the following way

$$ds^2 = \frac{l^2}{y^2}\left[-F(u,y,x^i)du^2 - 2dudv + dy^2 + \delta_{ij}dx^i dx^j\right], \tag{1}$$

where $i = 1, \dots, D-3$. In this case, the Einstein tensor of the above AdS wave metric $G_{\alpha\beta}$ and the one associated with the pp-wave metric inside of the square bracket, which we denote by $\bar{G}_{\alpha\beta}$, satisfy

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} \propto k_\alpha k_\beta, \qquad \bar{G}_{\alpha\beta} \propto k_\alpha k_\beta, \tag{2}$$

where $k^\mu\partial_\mu = \partial_v$ is a common null Killing vector and $\Lambda = -(D-2)(D-1)/(2l^2)$. These properties (2) imply interesting consequences if one considers the coupling of matter sources to these gravitational waves. Indeed, in both cases they indicate that the waves can only be supported by sources behaving as pure radiation fields [9], i.e. configurations for which all the components of the energy-momentum tensor vanish except the energy density along the retarded time u. For a generic matter field this condition imposes the fulfillment of *pure radiations constraints*, consisting in demanding the vanishing of all the components of the energy-momentum tensor except the quoted energy density. These conditions are strong requirements on the possible choices of matter sources to consider. Recently, we have been concerned by this problem in three dimensions as we were interested on the generation of exact gravitational waves propagating on flat space [10, 11, 12, 13] and AdS space [14], respectively. We have shown that self-gravitating scalar fields nonminimally coupled to these gravitational waves do not yield to inconsistencies, instead the pure radiations constraints have nontrivial solutions in these cases characterized by various interesting properties due to this particular coupling [14]. Among these curiosities, we have shown that the three-dimensional scalar source supporting an AdS wave [14] and the one compatible with the pp-wave [12] are conformally related in spite of the fact that their involved backgrounds are not necessarily related. In other words, this means that in both situations, the source is independent of the structural metric function F of Eq. (1), and in some sense is only sensitive to the general form of the spacetime metric. At our opinion, it seems interesting to explore wether the analysis performed in three dimensions is only due to the simplicity of $2+1$ gravity, or it is generic to any dimension. The main purpose of this paper is to carry out the correspondence mentioned above to higher dimensions by establishing a conformal mapping between the pure radiation constraints determining the scalar sources in each gravitational wave background.

The organization of the paper is the following. We first consider the problem of scalar fields nonminimally coupled to a pp-wave background in arbitrary dimension. The resulting pure radiation constraints are solved in full generality and it is shown that their integration fixes uniquely the matter source without the explicit knowledge of the structural metric function. In particular, there exists a unique self-interaction, depending on a single coupling constant, allowing the scalar field to act as a source of the pp-wave. In the third section, we reproduce the same analysis in the context of scalar fields nonminimally coupled to AdS waves. The pure radiation constraints are again integrable and the AdS wave scalar source is completely determined. In this case, the process singles out a unique self-interaction potential depending on two coupling constants. In the fourth section of the paper, we establish a conformal correspondence between the pure radiation constraints of both systems assuming the scalar fields of both backgrounds are conformally related. In other words, this means that starting from a pp-wave scalar field configuration one is able to generate the scalar field configuration supporting the AdS wave. In this correspondence, we emphasize the problem of the mismatching of the coupling constants of the potentials, problem which is only specific to higher dimensions. We provide a recipe which permits to obtain the exact potential supporting the AdS wave, with its two coupling constants, starting from the *pp*-wave one with its single coupling constant. In the last section, we summarize our results and leave some open questions related to this work. Two Appendixes are included at the end where the detailed higher-dimensional field equations on each background are explicitly written.

II. *PP*-WAVES SUPPORTED BY NONMINIMALLY COUPLED SCALAR FIELDS

In this first part, we are concerned with scalar field nonminimally coupled to a *pp*-wave background in D dimensions defined by the line element

$$\begin{aligned} d\bar{s}^2 &= -\bar{F}(u,x^{\hat{i}})du^2 - 2dudv + \delta_{\hat{i}\hat{j}}dx^{\hat{i}}dx^{\hat{j}}, \\ &= -\bar{F}(u,y,x^i)du^2 - 2dudv + dy^2 + \delta_{ij}dx^i dx^j, \end{aligned} \tag{3}$$

where the *plane* wave-fronts of the gravitational wave have coordinates $x^{\hat{i}} = (y, x^i)$ with $i = 1, \ldots, D-3$, and its *parallel* rays are described by the null covariantly constant field $k^\mu\partial_\mu = \partial_v$. The action we are concerned with is given by

$$\bar{S}(\bar{g}_{\alpha\beta}, \bar{\Phi}) = \int d^D x\, \sqrt{-\bar{g}} \left(\frac{1}{2\kappa}\bar{R} - \frac{1}{2}\bar{g}^{\alpha\beta}\nabla_\alpha\bar{\Phi}\nabla_\beta\bar{\Phi} - \frac{1}{2}\xi\bar{R}\,\bar{\Phi}^2 - \bar{U}(\bar{\Phi}) \right), \tag{4}$$

where ξ is the parameter characterizing the nonminimal coupling to gravity of the scalar field $\bar{\Phi}$, whose self-interaction potential is described by $\bar{U}(\bar{\Phi})$. For later convenience we have used the convention that all the bared quantities are those relatives to the *pp*-wave background. The field equations obtained by varying the action with respect to the metric and the scalar field read

$$\bar{G}_{\alpha\beta} = \kappa\bar{T}_{\alpha\beta}, \tag{5}$$

and

$$\bar{\Box}\bar{\Phi} = \xi\bar{R}\,\bar{\Phi} + \frac{\mathrm{d}\bar{U}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}, \tag{6}$$

respectively, where the energy-momentum tensor is given by

$$\bar{T}_{\alpha\beta} = \nabla_\alpha\bar{\Phi}\nabla_\beta\bar{\Phi} - \bar{g}_{\alpha\beta}\left(\frac{1}{2}\bar{g}^{\mu\nu}\nabla_\mu\bar{\Phi}\nabla_\nu\bar{\Phi} + \bar{U}(\bar{\Phi})\right) + \xi\left(\bar{g}_{\alpha\beta}\bar{\Box} - \bar{\nabla}_\alpha\bar{\nabla}_\beta + \bar{G}_{\alpha\beta}\right)\bar{\Phi}^2. \tag{7}$$

In order to study these configurations we assume that the null Killing field $k^\mu\partial_\mu = \partial_v$ is also a symmetry of the scalar field, i.e. $\bar{\Phi} = \bar{\Phi}(u, x^{\hat{i}})$. The independent Einstein equations on the *pp*-wave background are given in Appendix A. As it was stressed in the introduction, the structure of the Einstein tensor for the geometry (3) is sketched as

$$\bar{G}_{\alpha\beta} \propto k_\alpha k_\beta, \tag{8}$$

which in the coordinates adapted to k^μ means that the only nonvanishing component of the Einstein tensor is the one along the retarded time $\bar{G}_{uu}$. Hence, all the components of the energy-momentum tensor except T_{uu} must vanish by virtue of the Einstein equations, and this is interpreted as the scalar field must behave like a pure radiation field [9]. As we shall see below the integration of the resulting pure radiation constraints uniquely determines the matter source. Finally, the remaining independent Einstein equation, i.e. the one along the component uu (see Appendix A), allows to derive the structural metric function $\bar{F}$ in the expression (3).

The independent pure radiation constraints are given by the following combinations, as can be noticed from Appendix (A),

$$\bar{T}_{u\hat{i}} = 0, \tag{9a}$$

$$\bar{T}_{\hat{i}\hat{j}} + \delta_{\hat{i}\hat{j}}\bar{T}_{uv} = 0, \tag{9b}$$

$$\delta^{\hat{i}\hat{j}}\bar{T}_{\hat{i}\hat{j}} + (D-3)\bar{T}_{uv} = 0. \tag{9c}$$

As it also occurs in three dimensions [12], it is useful to make the following redefinition of the scalar field

$$\bar{\Phi} = \frac{1}{\bar{\sigma}^{2\xi/(1-4\xi)}}, \tag{10}$$

where we are considered all the possible values of the nonminimal coupling parameter except $\xi = 0$ and $\xi = 1/4$. These two cases can also be studied but their analysis is not essential for our main task. Using Eq. (A1) of Appendix A the first two equations (9a) and (9b) reduce to

$$\partial_{u\hat{i}}\bar{\sigma} = 0, \tag{11a}$$

$$\partial_{\hat{i}\hat{j}}\bar{\sigma} = 0, \tag{11b}$$

which imply that the general solution is separable in all the coordinates and is additionally linear in the planar coordinates of the wave-front

$$\bar{\sigma} = \bar{k}_{\hat{i}}x^{\hat{i}} + \bar{f}(u) \tag{12}$$

where the $\bar{k}_{\hat{i}}$ are $D-2$ arbitrary constants and $\bar{f}$ is an arbitrary function of the retarded time. Hence, the integration of the first two pure radiation constraints completely determines the scalar field. Inserting the obtained expression into the remaining pure radiation constraint (9c), the allowed potential is singled out as

$$\bar{U}(\bar{\Phi}) = \frac{2\xi^2\lambda}{(1-4\xi)^2}\bar{\Phi}^{(1-2\xi)/\xi}, \tag{13}$$

where we have defined the coupling constant by

$$\lambda = \delta^{\hat{i}\hat{j}}\bar{k}_{\hat{i}}\bar{k}_{\hat{j}}. \tag{14}$$

The emergence of such potential is interesting by itself for various reasons. Firstly, for the conformal coupling in D dimensions, $\xi = \xi_D = (D-2)/[4(D-1)]$, the above potential reduces to the only potential compatible with the conformal invariance in higher dimensions $\bar{U}(\bar{\Phi}) \propto \bar{\Phi}^{2D/(D-2)}$. We stress from now that the potential (13) depends on a single coupling constant λ expressed in terms of the $D-2$ integration constants of the scalar field. The solution of the pure radiation constraints also allows the existence of nontrivial free massless configurations for $\lambda = 0$, as it can be noticed from Eqs. (14) and (12). In this case, the free scalar field depends only and arbitrarily on the retarded time. It is also interesting to notice that for the particular value of the nonminimal parameter $\xi = 1/2$, the potential (13) reduces to a positive constant and hence, the original problem is equivalent to considering the Einstein equations with a positive effective cosmological constant in presence of a free nonminimally coupled scalar field.

For later convenience, we record that some of the pure radiation constraints (11) can be reexpressed compactly in terms of the scalar field and its allowed self-interaction potential (13) as

$$\partial_\alpha \left(\frac{\partial_y \bar{\Phi}}{\sqrt{\bar{U}\left(\bar{\Phi}\right)}} \right) = 0, \tag{15}$$

which in addition empathizes that the expression between the parenthesis is a constant. This condition can be interpreted as a sort of Bogomolnyi self-dual condition for the system.

In sum, we have seen that the integration of the pure radiation constraints have completely determined the matter source. It is interesting to stress that this task have been achieved without using the structural metric function $\bar{F}$. In other words, this means that the matter source is only sensitive to the form of the metric and not to the specific structural metric function. As we shall see below, this property is also present in the case of AdS waves. Hence, in order to relate the matter sources of both backgrounds, it is not necessary to derive the structural metric functions.

III. ADS WAVES SUPPORTED BY NONMINIMALLY COUPLED SCALAR FIELDS

In this section, we are concerned with scalar fields nonminimally coupled to an AdS wave

$$ds^2 = \frac{l^2}{y^2}\left[-F(u,y,x^i)du^2 - 2dudv + dy^2 + dx_i dx^i\right], \tag{16}$$

where the wave fronts $\{u, v = \text{const.}\}$ are now hyperboloids with curvature proportional to $-1/l^2$ and coordinates $x^{\hat{i}} = (y, x^i)$, $i = 1, \ldots, D-3$. The field equations are those arising from the following action

$$S = \int d^D x\, \sqrt{-g}\left(\frac{1}{2\kappa}(R + 2\Lambda) - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi - \frac{1}{2}\xi R\,\Phi^2 - U(\Phi)\right), \tag{17}$$

where $\Lambda = -(D-2)(D-1)/(2l^2)$ is the negative cosmological constant, ξ is the nonminimal coupling parameter, and $U(\Phi)$ is the self-interaction potential. The involved field equations are the Einstein and the nonlinear Klein–Gordon equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{18}$$

$$\Box\Phi = \xi R\,\Phi + \frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}, \tag{19}$$

where the corresponding energy-momentum tensor is defined by

$$T_{\alpha\beta} = \nabla_\alpha\Phi\nabla_\beta\Phi - g_{\alpha\beta}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + U(\Phi)\right) + \xi\left(g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta + G_{\alpha\beta}\right)\Phi^2. \tag{20}$$

We now see that the strategy used in the case of the *pp*-wave background can be exactly reproduced here in order to determine the allowed matter source for an AdS wave background. The clue lies in the fact that the Einstein tensor for an AdS wave background has the following structure

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} \propto k_\alpha k_\beta, \tag{21}$$

with $k^\mu \partial_\mu = \partial_v$, which implies that any self-gravitating source supporting the wave in the presence of the negative cosmological constant must behave effectively as a pure radiation field [9]. As a consequence, in the coordinates of metric (16) the only component of the Einstein equations (18) with a nonvanishing left hand side is the uu one as it occurs in the *pp*-wave background. The other Einstein equations reduce again to pure radiation constraints.

We assume that the null Killing field k^μ is also a symmetry of the scalar field which in turn implies that the independent field equations on the AdS wave background reduce to the ones given in Appendix B. As before, the pure radiation constraints are expressed as

$$\begin{aligned} T_{u\hat{i}} &= 0, & \text{(22a)}\\ T_{\hat{i}\hat{j}} + T_{uv}\delta_{\hat{i}\hat{j}} &= 0, & \text{(22b)}\\ \delta^{\hat{i}\hat{j}} T_{\hat{i}\hat{j}} + (D-3)T_{uv} &= 0. & \text{(22c)} \end{aligned}$$

We now consider the following redefinition for the scalar field

$$\Phi = \frac{1}{\sigma^{2\xi/(1-4\xi)}}, \tag{23}$$

which presents the advantage that the pure radiation constraints (22a) and (22b), whose explicit form can be found in the Eq. (B1) of Appendix B, are more simple to tackle

$$\begin{aligned} \partial_y\left(y\partial_u\sigma\right) &= 0, & \text{(24a)}\\ \partial^2_{ui}\sigma &= 0, & \text{(24b)}\\ \partial_y\left(y^2\partial_y\sigma\right) &= 0, & \text{(24c)}\\ \partial_y\left(y\partial_i\sigma\right) &= 0, & \text{(24d)}\\ \partial^2_{ij}\sigma &= 0. & \text{(24e)} \end{aligned}$$

The general solution of (24) is given by

$$\sigma(u,y,x^i) = \frac{l}{y}\left[k_y y + k_i x^i + f(u)\right], \tag{25}$$

where k_y and k_i are $D-2$ integration constants and f is a general function of the retarded time. The remaining pure radiation constraint (22c) is the one that permits to obtain the allowed potential. After a tedious but straightforward calculation we conclude that the only self-interaction potential allowed by the system is given by

$$\begin{aligned} U(\Phi) \;=\; & \frac{2\xi\Phi^2}{(1-4\xi)^2}\Bigg(\xi\lambda_1\Phi^{(1-4\xi)/\xi} - 8(D-1)\xi(\xi-\xi_D)\lambda_2\Phi^{(1-4\xi)/(2\xi)} \\ & + \frac{4D(D-1)}{l^2}(\xi-\xi_D)(\xi-\xi_{D+1})\Bigg), \end{aligned} \tag{26}$$

where $\xi_D = (D-2)/[4(D-1)]$ is the conformal coupling in dimension D and the two coupling constants of the potentials are defined by

$$\lambda_1 = {k_y}^2 + \delta^{ij} k_i k_j, \qquad \lambda_2 = \frac{k_y}{l}. \tag{27}$$

Once again, the emergence of such potential is intriguing for various reasons. In contrast with the allowed potential in the *pp*-wave situation (13), the AdS wave potential (26) depends on two coupling constants (27). This subtlety is not present in three dimensions [14] since in this case $k_i = 0$, and hence the coupling constants are related, i.e. $\lambda_2 = \sqrt{\lambda_1}/l$. This remark will be of importance in the next section where a correspondence between the two configurations previously analyzed will be presented. It is also interesting to note that for the conformal value of the nonminimal coupling parameter, $\xi = \xi_D$, the expression (26) also reduces to the conformally invariant potential as it occurs in the *pp*-wave case. In fact, at the vanishing cosmological constant limit ($l \to \infty$), we recover the potential permitted by the *pp*-wave background (13). Finally, we also mention that this potential is exactly the one arising in the context of scalar fields nonminimally coupled to special geometries without inducing backreaction (the static BTZ black hole [15, 16], flat space [17, 18], and the generalized (A)dS spacetimes [19]). All these examples share a common feature, namely the existence of nontrivial solutions with a vanishing energy-momentum tensor called stealth configurations.

IV. THE CORRESPONDENCE

In this section, we establish a correspondence between the two sets of pure radiations constraints previously studied. The existence of a map between the involved sources was first noticed in three dimension [14]. Here, we prove that this equivalence is not a mere consequence of the apparent simplicity of $2+1$ gravity and in fact, it can be extended to higher dimensions. In a more precise set-up, assuming a conformal relation between the scalar fields generating the two gravitational waves, we first put in relation the pure radiation constraints that determine the scalar field solution in both systems. The remaining pure radiation constraint is the one that fixes the allowed potential of each system. The relation between these last two constraints is studied in the second part because of the subtlety due to the mismatching of the coupling constants of both potentials.

The functional expressions for both scalar fields, on the one hand Eqs. (10) and (12), and on the other hand Eqs. (23) and (25), suggest to consider a conformal relation between them in the following manner

$$\Phi = \left(\frac{l}{y}\right)^s \bar{\Phi}, \tag{28}$$

where the conformal weight s is not fixed *ab initio*. Using this relation we intent to write the pure radiation constraints resulting from an AdS wave (22) in terms of the ones implied by the existence of a *pp*-wave (9). Running down the components list of the first two sets of pure radiation constraints (the ones that fix the scalar field dependence) we obtain the following relations

$$T_{uy} = \left(\frac{l}{y}\right)^{2s}\left(\bar{T}_{uy} - [s(1-4\xi)+2\xi]\,\frac{\partial_u\bar{\Phi}^2}{2y}\right), \tag{29a}$$

$$T_{ui} = \left(\frac{l}{y}\right)^{2s}\bar{T}_{ui}, \tag{29b}$$

$$T_{yy}+T_{uv} = \left(\frac{l}{y}\right)^{2s}\left[\bar{T}_{yy}+\bar{T}_{uv} - [s(1-4\xi)+2\xi]\,y^{s-1}\partial_y\left(\frac{\bar{\Phi}^2}{y^s}\right)\right], \tag{29c}$$

$$T_{yi} = \left(\frac{l}{y}\right)^{2s}\left(\bar{T}_{yi} - [s(1-4\xi)+2\xi]\,\frac{\partial_i\bar{\Phi}^2}{2y}\right), \tag{29d}$$

$$T_{ij}+\delta_{ij}T_{uv} = \left(\frac{l}{y}\right)^{2s}\left(\bar{T}_{ij}+\delta_{ij}\bar{T}_{uv}\right). \tag{29e}$$

It is clear from these relations that the particular value of the conformal weight

$$s = -\frac{2\xi}{1-4\xi}, \tag{30}$$

seems to play a crucial role, but since it remains to connect the pure radiation constraints (9c) and (22c), we prefer to keep the weight infixed for now. The possible relation between these two remaining constraints is more subtle, since this would imply an interrelation among the potentials (13) and (26). As it has been pointed out previously, on the *pp*-wave side there is only one coupling constant λ, in contrast with the AdS wave case where two a priori independent coupling constants λ_1 and λ_2 are present. Hence, in order to establish the correspondence we need to provide a recipe for choosing the two coupling constants of the AdS wave source starting from the *pp*-wave one. This problem does not appear in $2+1$ dimensions where there is only one wave-front coordinate and only one related integration constant, giving rise to a single coupling constant for both gravitational wave backgrounds [14]. In order to compensate this mismatch, our first election is simple and consists of choosing λ_1 coinciding with the single coupling constant of the potential supporting the *pp*-wave. The second election is inspired by the self-dual condition (15) which implies that the quantity $(\partial_y\bar{\Phi})/\bar{U}(\bar{\Phi})^{1/2}$ is constant for the *pp*-wave configuration, and hence this allows us to define the coupling constant λ_2 as proportional to this constant. Using the following two definitions for the coupling constants

$$\lambda_1 = \lambda, \tag{31a}$$

$$\lambda_2 = -\frac{1}{l}\sqrt{\frac{\lambda}{2}}\frac{\partial_y\bar{\Phi}}{\sqrt{\bar{U}\left(\bar{\Phi}\right)}}, \tag{31b}$$

we conclude, after a tedious computation, that the remaining pure radiation constraints are related as follows

$$\delta^{\hat{i}\hat{j}}T_{\hat{i}\hat{j}}+(D-3)T_{uv}+\frac{l^2}{y^2}\left[U(\Phi)-V_s(\Phi,y)\right] = \left(\frac{l}{y}\right)^{2s}\left(\delta^{\hat{i}\hat{j}}\bar{T}_{\hat{i}\hat{j}}+(D-3)\bar{T}_{uv}\right), \tag{32}$$

where the function V_s depends on the scalar field Φ and additionally on the wave-front coordinate y by means of

$$
\begin{aligned}
V_s(\Phi,y) \;=\; & \frac{\lambda_1}{\lambda}\bar{U}(\Phi)\left(\frac{l}{y}\right)^{[s(4\xi-1)-2\xi]/\xi} + [s+2\xi(D-1)]\frac{\lambda_2}{\sqrt{\lambda}}\Phi\sqrt{2\bar{U}(\Phi)}\left(\frac{l}{y}\right)^{[s(4\xi-1)-2\xi]/(2\xi)} \\
& + \frac{[s^2+4(D-1)\xi s+(D-2)(D-1)\xi]}{2l^2}\Phi^2,
\end{aligned} \tag{33}
$$

where $\bar{U}(\Phi)$ stands for the functional dependence of the *pp*-wave potential (13) evaluated on the AdS wave scalar field.

We are now in position to derive some conclusions. Firstly, as it was previously mentioned it is clear from the relations (29) that the involved pure radiation constrains of both gravitational wave backgrounds are conformally related only if one choose the conformal weight (30). Secondly, for such weight the function V_s above becomes y-independent and reduces precisely to the functional expression of the scalar potential supporting the AdS wave (26). Hence, this process automatically select the indicated potentials as the only ones allowing the conformal mapping between the pure radiation constraints! It is also interesting to note that for the conformal coupling in D dimensions, $\xi=\xi_D$, the weight (30) becomes $s=(2-D)/2$ which is precisely the conformal weight associated to the conformal Klein–Gordon equation in D dimensions.

In summary, we have shown that the pure radiation constraints on a *pp*-wave and an AdS wave are conformally related if one suppose a conformal relation (28) between the involved scalar fields with a conformal weight (30), and additionally the respective potential (13) and (26) are taken in each side using the definitions (31) for the coupling constants, i.e.

$$T_{u\hat{i}} \;=\; \left(\frac{l}{y}\right)^{-4\xi/(1-4\xi)}\bar{T}_{u\hat{i}}, \tag{34a}$$

$$T_{\hat{i}\hat{j}}+\delta_{\hat{i}\hat{j}}T_{uv} \;=\; \left(\frac{l}{y}\right)^{-4\xi/(1-4\xi)}\left(\bar{T}_{\hat{i}\hat{j}}+\delta_{\hat{i}\hat{j}}\bar{T}_{uv}\right), \tag{34b}$$

$$\delta^{\hat{i}\hat{j}}T_{\hat{i}\hat{j}}+(D-3)T_{uv} \;=\; \left(\frac{l}{y}\right)^{-4\xi/(1-4\xi)}\left(\delta^{\hat{i}\hat{j}}\bar{T}_{\hat{i}\hat{j}}+(D-3)\bar{T}_{uv}\right). \tag{34c}$$

A similar conclusion can be achieved by studying the wave equation for the scalar field. Using the conformal weight (30) and the relations (31) between the coupling constants of both potentials, a conformal relation between the Klein–Gordon equations can be also achieved for any generic nonminimal coupling ξ,

$$\Box\Phi-\xi R\,\Phi-\frac{\mathrm{d}U(\Phi)}{\mathrm{d}\Phi}=\left(\frac{l}{y}\right)^{2(2\xi-1)/(1-4\xi)}\left(\bar{\Box}\bar{\Phi}-\xi\bar{R}\,\bar{\Phi}-\frac{\mathrm{d}\bar{U}(\bar{\Phi})}{\mathrm{d}\bar{\Phi}}\right). \tag{35}$$

This relation is far from obvious since it usually only works in the case of the conformal coupling and taking in both sides of the equation the unique potential that does not spoil the conformal invariance. A fact which is straightforwardly recovered in the above expression just taking $\xi=\xi_D$.

V. CONCLUSIONS

Here, we have been concerned with the *pp*-wave and the AdS wave backgrounds in arbitrary dimension. These spacetimes share in common that their coupling to a matter source is accompanied by a strong restriction, namely the source field must behave like a pure radiation field. The elaboration of this work through two concise examples in arbitrary dimensions has opened a number of questions that we would like to comment.

In this paper, we have shown that the nonminimal coupling of scalar fields to these particular spacetimes can be realized consistently, and the most general scalar field configurations consistent with the only symmetry of the problem have been derived. In this first result there is an interesting contrast between the strong restriction imposed by the spacetimes and the fact that the most general solution of an higher-dimensional problem with only one symmetry can be obtained. Moreover, it is obvious form our study that not any matter field can act as a source for these backgrounds. In view of this, it is legitimate to go into thoroughly and ask what are the characteristics that a matter action must possess in order to couple consistently with these peculiar spacetimes. For example, it is clear that since the *pp*-wave and the AdS wave metrics possess a null Killing field together with the fact that their Einstein tensors have the structure (2), automatically impose an on-shell traceless condition on the energy-momentum tensor of the matter source.

In the analysis of the pure radiation constraints, we have put in evidence the analogies existing between both backgrounds. Indeed, in each case, the same combinations of the energy-momentum tensor components give rise to the independent pure radiation constraints. Moreover, these combinations uniquely fix the scalar field source, that means not only the local expression for the scalar field but also the unique self-interactions allowing the existence of the whole configuration. Furthermore, this derivation has been done without the explicit knowledge of the structural metric function, suggesting that the source is only sensitive to the general form of the metric. This property by itself is very intriguing and unusual in gravitational physics due to the strongly coupled behavior inherent to the matter/gravity interaction; as it is well known matter acts as the source of spacetime curvature generating the gravitational potential, but at the same time the spacetime geometry is the arena where matter fields evolve, i.e. matter fields feel the fingerprints of the gravitational field via its equation of motion. In the present cases the metric structural functions do not participate in the Klein–Gordon equations. It is natural to ask first wether there exist other examples of such behavior in the current literature. To our knowledge the only similar examples occur for the so-called stealth configurations for which both matter and gravity are completely decoupled [15, 16, 17, 19, 20]. The analogies with the stealth configurations also concern the allowed potentials as it has been stressed in the present work. For all these reasons, it would be desirable to have a better understanding of these curiosities from a mathematical as well as physical point of view.

As said before, the pure radiation constraints impose and single out a unique form of the potential for each background. In the *pp*-wave case, the selected self-interaction depends on a single coupling constant and follows a power-law dependence on the scalar field. In the AdS wave case, two coupling constants emerge from the integration of the pure radiation constraints, each one associated to a different power-law term in the potential, additionally a third contribution also appears consisting in a mass term whose mass scale is fixed by the AdS radius. In spite of being different potentials, in the case of the conformal coupling in D dimensions, both potentials reduce to the conformally invariant potential. It is appealing that as the nonminimal coupling parameter takes the conformal value, the allowed potentials precisely reduce to the conformally invariant one in D dimensions. This may be think as if for an arbitrary value of the nonminimal coupling parameter, the system would enjoy a symmetry higher than the conformal symmetry and reduces to this later as the nonminimal coupling parameter becomes the conformal one. The conformal relations established in the previous section brings evidence in favor of this view. An interesting work will then consist of studying the dynamical symmetries of the models we have considered in order to confirm the existence or not of a higher symmetry including the conformal one.

In the last part, we have extended the analogies observed between the AdS wave and the *pp*-wave sources by establishing a conformal correspondence between the pure radiation constraints of each system. In some sense, this correspondence permits to derive the scalar field configuration of the AdS wave background from the *pp*-wave one in a nontrivial way. In this correspondence, the scalar fields are conformally related with a weight expressed in terms of the nonminimal coupling parameter independently of the precise dimension. For the conformal value of the nonminimal coupling, this weight precisely becomes the conformal weight associated to the conformal Klein–Gordon equation. The pure radiation constraints fixing the scalar field dependence are easily put in equivalence in contrast with the radiation constraints that single out the self-interactions. Indeed, in this last case, there is a mismatching between the coupling constants of the respective potentials. The additional coupling constant in the AdS wave potential has been shown to be associated to a constant arising from a self-dual condition naturally satisfied by the *pp*-wave scalar fields. It is far from obvious that the self-gravitating matter sources generating each backgrounds are in correspondence even if these backgrounds can be viewed as conformally related. One may think that the correspondence established here is a sort of residual conformal symmetry that has its origin on the on-shell traceless condition of the energy-momentum tensor, a property usually associated to the conformal invariance of the source. Once again, it would be of interest to understand the mathematical structures that are behind of the examples treated in this work.

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APPENDIX A: FIELD EQUATIONS FOR HIGHER DIMENSIONAL *PP*-WAVES

The independent Einstein equations (18) for the energy-momentum (20) on the background of a D-dimensional *pp*-wave (3) are given by the following combination

$$\begin{aligned} 0 &= \bar{G}_{\alpha\beta} - \kappa \bar{T}_{\alpha\beta} + \bar{g}_{\alpha\beta}(\bar{G}_{uv} - \kappa \bar{T}_{uv}) \\ &= \left[\frac{1}{2}(1-\kappa\xi\bar{\Phi}^2)\hat{\triangle}\bar{F} + \kappa\xi\left(\partial^2_{uu}\bar{\Phi}^2 - \frac{1}{2}\delta^{\hat{i}\hat{j}}\partial_{\hat{i}}\bar{F}\partial_{\hat{j}}\bar{\Phi}^2\right) - \kappa(\partial_u\bar{\Phi})^2\right]\delta^u_\alpha\delta^u_\beta \\ &\quad - 2\kappa\left(\partial_u\bar{\Phi}\partial_{\hat{i}}\bar{\Phi} - \xi\partial^2_{u\hat{i}}\bar{\Phi}^2\right)\delta^u_{(\alpha}\delta^{\hat{i}}_{\beta)} - \kappa\left(\partial_{\hat{i}}\bar{\Phi}\partial_{\hat{j}}\bar{\Phi} - \xi\partial^2_{\hat{i}\hat{j}}\bar{\Phi}^2\right)\delta^{\hat{i}}_\alpha\delta^{\hat{j}}_\beta, \end{aligned} \tag{A1}$$

and the trace

$$0=\bar{g}^{\hat{i}\hat{j}}(\bar{G}_{\hat{i}\hat{j}}-\kappa\bar{T}_{\hat{i}\hat{j}})+(D-3)(\bar{G}_{uv}-\kappa\bar{T}_{uv})=\kappa\left(\bar{U}(\bar{\Phi})-\frac{1}{2}\delta^{\hat{i}\hat{j}}\partial_{\hat{i}}\bar{\Phi}\partial_{\hat{j}}\bar{\Phi}\right), \tag{A2}$$

where $x^{\hat{i}}=(y,x^i)$, $i=1,\ldots,D-3$, and $\hat{\triangle}=\delta^{\hat{i}\hat{j}}\partial_{\hat{i}}\partial_{\hat{j}}$. It is straightforward to check that the above equations reduce to its $2+1$ dimensional versions solved in Ref. [12].

APPENDIX B: FIELD EQUATIONS FOR HIGHER DIMENSIONAL ADS WAVES

All the information following from Einstein equations (18) with energy-momentum (20) on the background of a D-dimensional AdS wave (16) is encoded in the following combination

$$\begin{aligned}
0 &= G_{\alpha\beta}+\Lambda g_{\alpha\beta}-\kappa T_{\alpha\beta}+\frac{y^2}{l^2}g_{\alpha\beta}(G_{uv}+\Lambda g_{uv}-\kappa T_{uv})\\
&= \left[\frac{1}{2}\frac{l^2}{y^2}(1-\kappa\xi\Phi^2)\Box F+\kappa\xi\left(\partial^2_{uu}\Phi^2-\frac{1}{2}\delta^{\hat{i}\hat{j}}\partial_{\hat{i}}F\partial_{\hat{j}}\Phi^2\right)-\kappa(\partial_u\Phi)^2\right]\delta^u_\alpha\delta^u_\beta\\
&\quad -2\kappa\left(\partial_u\Phi\partial_y\Phi-\frac{\xi}{y}\partial_y\left(y\partial_u\Phi^2\right)\right)\delta^u_{(\alpha}\delta^y_{\beta)}-2\kappa\left(\partial_u\Phi\partial_i\Phi-\xi\partial^2_{ui}\Phi^2\right)\delta^u_{(\alpha}\delta^i_{\beta)}\\
&\quad -\kappa\left((\partial_y\Phi)^2-\frac{\xi}{y^2}\partial_y\left(y^2\partial_y\Phi^2\right)\right)\delta^y_\alpha\delta^y_\beta-2\kappa\left(\partial_y\Phi\partial_i\Phi-\frac{\xi}{y}\partial_y\left(y\partial_i\Phi^2\right)\right)\delta^y_{(\alpha}\delta^i_{\beta)}\\
&\quad -\kappa\left(\partial_i\Phi\partial_j\Phi-\xi\partial^2_{ij}\Phi^2\right)\delta^i_\alpha\delta^j_\beta,
\end{aligned} \tag{B1}$$

and the trace

$$\begin{aligned}
0 &= g^{\hat{i}\hat{j}}(G_{\hat{i}\hat{j}}+\Lambda g_{\hat{i}\hat{j}}-\kappa T_{\hat{i}\hat{j}})+(D-3)\frac{y^2}{l^2}(G_{uv}+\Lambda g_{uv}-\kappa T_{uv})\\
&= \kappa\left(U(\Phi)+\xi\Lambda\Phi^2+\xi(D-1)\frac{y}{l^2}\partial_y\Phi^2-\frac{1}{2}\frac{y^2}{l^2}\delta^{\hat{i}\hat{j}}\partial_{\hat{i}}\Phi\partial_{\hat{j}}\Phi\right),
\end{aligned} \tag{B2}$$

where $x^{\hat{i}}=(y,x^i)$, $i=1,\ldots,D-3$, $\Lambda=-(D-2)(D-1)/(2l^2)$, and

$$\Box F=\frac{y^2}{l^2}\left[y^{D-2}\partial_y\left(\frac{1}{y^{D-2}}\partial_y F\right)+\triangle F\right], \tag{B3}$$

with $\triangle=\delta^{ij}\partial_i\partial_j$. As before the above equations becomes the $2+1$ dimensional ones studied in Ref. [14].

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